1 (Sem–5/FYUGP) MAT 42 MJ

2025

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Date
Nagaon Assam

MATHEMATICS

(Major)

Paper : MAT0500204

(Theory of Real Functions)

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×8=8

(a) Is every limit point of a set $A \subseteq R$ necessarily an element of A?

(b) Write the ε-δ definition for

$$\lim_{x\to 3}(2x) = 6$$

(c) A function f is continuous at a if whenever a sequence $\{x_n\}$ gets closer to a, what must happen to the sequence $\{f(x_n)\}$?

(d) If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, what must happen according to the intermediate value theorem?

(e) Is every continuous function on a closed and bounded interval $[a, b]$ uniformly continuous?

(f) If f' exists on $[a, b]$, what must be true about f' on the interval $[a, b]$?

(g) Taylor's theorem can be used to approximate functions. If $R_n(x)$ is the remainder term of order n, what inequality does it satisfy for some ξ between 0 and x?

(h) A function f is differentiable at $x = a$ if and only if there exists a function $\phi(x)$ continuous at a such that $f(x) = f(a) + \phi(x)(x - a)$. What is $\phi(a)$?

2. Answer any *six* of the following questions : 2×6=12

(a) Let $A = \{\frac{1}{n} : n \in N\}$.

(i) Find all limit points of A;

(ii) Is 0 an element of A?

(b) Using ε-δ definition prove that

$$\lim_{x\to 2}(3x+1) = 7$$

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(c) Evaluate :

(i) $\lim_{x \to 0+} \frac{1}{x^2}$

(ii) $\lim_{x \to -\infty} \frac{2x^2+3}{x^2-1}$

(d) Let $f(x) = x^2$. Using sequence, show that f is continuous at $x = 3$.

(e) Let $f(x) = x^3 - 3x$ on $[-2, 2]$. Find the maximum and minimum values of f on this interval.

(f) Let $f(x) = x^3 - x - 2$, show that f has a root in the interval $[1, 2]$.

(g) Let $f(x) = x^2$. Using Caratheodory's approach, write $f(x)$ in the form $f(x) = f(a) + \phi(x)(x-a)$ and find $\phi(a)$ at $a = 1$.

(h) If $y = \sin(x^2)$, find $\frac{dy}{dx}$ using the chain rule.

(i) Find the first four terms of the Taylor series of $f(x) = \frac{1}{1+x}$ about $x = 0$.

(j) Suppose f is differentiable on $[0, 2]$ with $f'(0) = 1$ and $f'(2) = 5$. What can you say about the values of f' on $(0, 2)$?

3. Answer any *four* of the following questions : 5×4=20

(a) Let

$$A = \left\{ \frac{n}{n+1} : n \in N \right\}$$

(i) Find all limit points of the set A.

(ii) Using the sequential criterion, prove that $\lim_{x \to 1} f(x) = 2$, for $f(x) = 2x$.

(iii) Is 1 an element of A? Justify.

1+2+2=5

(b) (i) Using the ε-δ definition, prove that

$$\lim_{x \to 3} (5x - 4) = 11$$

(ii) Evaluate the one-sided limit

$$\lim_{x \to 0+} \frac{1}{x}$$

(iii) Find $\lim_{x \to \infty} \frac{3x^2+2}{x^2-1}$.

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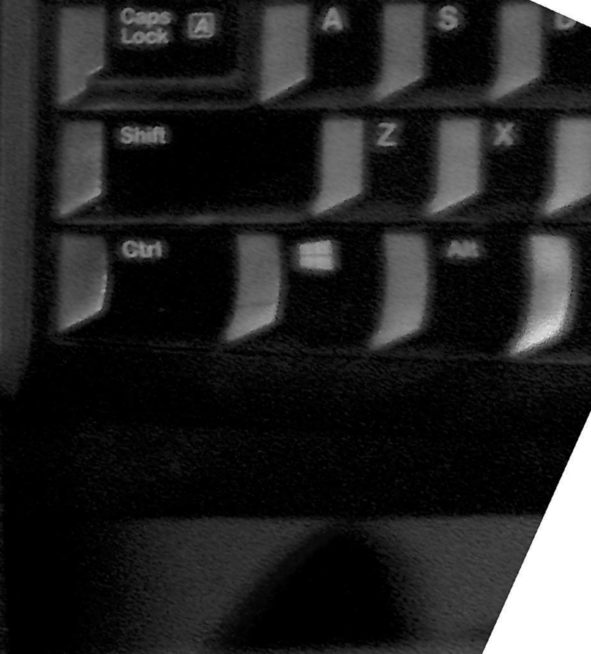
(c) Let $f(x) = x^2$ and $g(x) = \sin x$.

(i) Using the sequential criterion, prove that f is continuous at $x = 2$

(ii) Using continuity of f and g, prove that $f(x) + g(x)$ is continuous at $x = 2$.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(d) Let $f(x) = x^3 - 6x^2 + 9x + 2$ on the interval [0, 4].

(i) Find the points where $f(x)$ attains its maximum and minimum on [0, 4] and determine the corresponding values.

(ii) Show that $f(x)$ has at least one root in the interval [0, 1].

(e) (i) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on [0, 4].

(ii) If f is strictly increasing and continuous on $[a, b]$, show that f^{-1} exists and is continuous on $[f(a), f(b)]$.

(f) Let $f(x) = \ln(1+x)$.

(i) Show that f is differentiable at $x = 1$.

(ii) Using Caratheodory's theorem, express $f(x) = f(1) + \phi(x)(x-1)$ and find $\phi(1)$.

(iii) Verify that $\phi(1) = f'(1)$.

(g) Let $f(x) = x^3 - 3x$ on $[-\sqrt{3}, \sqrt{3}]$.

(i) Verify that f satisfies the conditions of Rolle's theorem.

(ii) Find all points c in $(-\sqrt{3}, \sqrt{3})$ where $f'(c) = 0$.

(iii) Apply the mean value theorem for the interval [−1, 2] to find c, such that $f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$.

(h) (i) Find the first four non-zero terms of the Taylor series of $\sin x$ about $x = 0$.

(ii) Find the first three non-zero terms of the Taylor's series of $\frac{1}{1+x}$ about $x = 0$.

(iii) Use the first two terms of the expansion of $(1+x)^5$, to approximate $(1 \cdot 02)^5$.

4. Answer any *two* of the following questions : 10×2=20

(a) Let $f : R - \{2\} \to R$ be defined by

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & , \ x \neq 2 \\ \text{undefined} & , \ x = 2 \end{cases}$$

(i) Show that 2 is a limit point of the domain.

(ii) Using the ε-δ definition of limit, prove that

$$\lim_{x \to 2} f(x) = 4$$

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(iii) Determine whether

$$\lim_{x\to 2-} f(x) \text{ and } \lim_{x\to 2+} f(x)$$

exist. Justify your answer.

(iv) Evaluate

$$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to\infty}(2x - f(x))$$

using appropriate limit theorems and state whether either limit diverges to infinity, diverges without limit or converges.

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(b) Let $f:[0, 3]\to R$ be defined by

$$f(x)=\begin{cases} x^2-2x, & 0\le x\le 2 \\ 4-x\ , & 2< x\le 3\end{cases}$$

(i) Using the sequential criterion determine whether f is continuous at $x=2$.

(ii) On the interval $[0, 3]$ verify that the maximum-minimum theorem holds and hence determine $\max f$ and $\min f$.

(iii) Show that the equation $f(x)=1$ has at least one solution in $[0, 3]$.

(iv) State with justification whether f is uniformly continuous on $[0, 3]$.

(Turn Over)

(c) Let $g:(1, 4]\to R$ be defined by $g(x)=\sqrt{x-1}$.

(i) Prove that g is continuous on its domain.

(ii) Determine whether g is uniformly continuous on $(1, 4]$. Give a justification using the uniform continuity.

(iii) Let $h(x)=x^3$. Show that since h is continuous and monotone on $\mathbb{R}$, its inverse h^{-1} is also continuous.

(iv) If $k(x)=g(x)+h(x)$, discuss whether k is continuous on $(1, 4]$.

(d) Let $f(x)=xe^x$, $x\in R$.

(i) Show that f is differentiable at all $x\in R$ and find $f'(x)$.

(ii) Let $g(x)=\ln(1+x)$, $x>-1$. Compute $\frac{d}{dx}g(f(x))$.

(iii) Using Rolle's theorem, show that there exists at least one $c\in(0, 1)$ such that

$$f'(c)=\frac{f(1)-f(0)}{1-0}$$

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(Continued)

(iv) Using Taylor's theorem, derive an inequality for e^x on [0, 1] of the form

$$1+x+\frac{x^2}{2} \le e^x \le 1+x+\frac{x^2}{2}+\frac{e}{6}x^3$$

(v) Using derivative of the inverse function, find $\frac{d}{dx}(f^{-1}(x))$ at $x = f(0)$.

(e) Let $h(x) = \ln(1+x)$, $x > -1$.

(i) Show that h is differentiable and find $h'(x)$.

(ii) Let $k(x) = e^x - 1$, $x \in R$. Using derivative of the inverse function, find $\frac{d}{dx}k^{-1}(x)$.

(iii) Using the Cauchy mean value theorem to show that $f(x) = \ln(1+x)$ and $g(x) = x$, there exists $c \in (0, a)$, such that

$$\frac{f(a)-f(0)}{g(a)-g(0)} = f'(c)$$

(iv) State Caratheodory's theorem in R^n. Then using the theorem, explain why any point in the convex hull of a set $S \subset R^3$ can be written as a convex combination of at most four points.

(v) If $f'(x_0) < 0$ and $f'(x_1) > 0$ for some differentiable function f, use Darboux's theorem to explain why there must exist a point c between x_0 and x_1 such that $f'(c) = 0$.

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