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3 (Sem-5/CBCS) STA HC 1

2022

STATISTICS

(Honours)

Paper : STA-HC-5016

(Stochastic Processes and Queueing Theory)

Full Marks : 60

Time : Three hours

**The figures in the margin indicate
full marks for the questions.**

1. Answer **any seven** of the following questions as directed : $1 \times 7 = 7$

(a) The value of $P(1)$ is

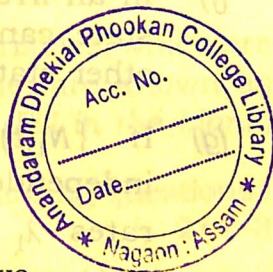
(i) 0

(ii) 1

(iii) ∞

(iv) None of the above

(Choose the correct option)



Contd.

(b) The mean of X in terms of the probability generating function (p.g.f.) of X is given by

(i) $P''(1)$

(ii) $P'(s)$

(iii) $P'(1)$

(iv) $P'(0)$

(Choose the correct option)

(c) The p.g.f. of sum of two independent random variables X and Y is the sum of the p.g.f. of X and that of Y .

(State True or False)

(d) Define state space of a stochastic process.

(e) A process which is not stationary is said to be _____. (Fill in the blank)

(f) In an irreducible Markov chain, every state cannot be reached from every other state. (State True or False)

(g) If $\{N_1(t)\}$ and $\{N_2(t)\}$ are two independent Poisson processes with rates λ_1 and λ_2 respectively then $N_1(t) - N_2(t)$ is a

(i) Poisson process with rate $\lambda_1 + \lambda_2$

(ii) Poisson process with rate $\lambda_1 - \lambda_2$

(iii) Poisson process with rate λ_1/λ_2

(iv) Not a Poisson process

(Choose the correct option)

(h) A state of a Markov chain is said to be ergodic if it is

(i) persistent non-null and aperiodic state

(ii) transient non-null and aperiodic state

(iii) persistent non-null and periodic state

(iv) transient null and aperiodic state

(Choose the correct option)

(i) Define traffic intensity.

(j) In M/M/1 queueing model, the interarrival time as well as service time follows _____ distribution.

(Fill in the blank)

(k) Define homogeneous Markov chain.

(l) Families of random variables, which are functions of, say, time, are known as _____. (Fill in the blank)

Answer **any four** of the following questions :
2×4=8

(a) Define bivariate probability generating function of a pair of random variables X and Y .

- (l) Families of random variables, which are functions of, say, time, are known as _____.
(Fill in the blank)

2. Answer **any four** of the following questions :
 $2 \times 4 = 8$

- (a) Define bivariate probability generating function of a pair of random variables X and Y .

- (b) Define transition probability matrix.
- (c) State *any two* postulates of Poisson process.
- (d) Is Poisson process a stationary process? If not, why?
- (e) Differentiate between steady state and transient state of a queueing system.
- (f) Distinguish between irreducible and reducible Markov chain.
- (g) What are the basic features of a queueing system?
- (h) Write *any two* properties of Poisson process.

3. Answer **any three** of the following questions:
5×3=15

- (a) Let X be a random variable with p.m.f
 $p_k = P_r\{X = k\} = q^k P, \quad k = 0, 1, 2, \dots$

$$0 < q = 1 - p < 1$$

Find the probability generating function (p.g.f.) of X and also find the mean and variance of X using probability generating function (p.g.f.) of X .



- (b) Define a stationary process.

Consider the process $X(t) = A_1 + A_2 t$
 where A_1, A_2 are independent random variables with $E(A_i) = a_i, \text{Var}(A_i) = \sigma_i^2, i = 1, 2$. Show that the process is not stationary.
 2+3=5

- (c) Let $\{X_n, n \geq 0\}$ be a Markov chain with three states 0, 1, 2 and with transition matrix

$$\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \text{ and initial distribution } P\{X_0 = i\} = 1/3, i = 0, 1, 2$$

$$\text{Find } P\{X_2 = 2/X_1 = 1\}$$

$$P\{X_2 = 2, X_1 = 1/X_0 = 2\}$$

$$P\{X_2 = 2, X_1 = 1, X_0 = 2\}$$

$$1+2+2=5$$

- (d) Define periodicity of the states of a Markov chain.

Consider the Markov chain with states 0, 1, 2 having transition matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that the states of the chain are periodic with period 2. $1+4=5$

- (e) If $\{N(t)\}$ is a Poisson process then prove that the auto-correlation coefficient between $N(t)$ and $N(t+s)$ is $\{t/(t+s)\}^{1/2}$.

- (f) The North-Eastern states of India are highly prone to earthquakes. Let us suppose that earthquakes occur at the rate of 2 per year, then

- (i) Find the probability that at least 3 earthquakes occur during the next two years.

- (ii) Find the probability distribution of the time, till the next quake.

$$2^{1/2} \times 2^{1/2} = 5$$

- (g) Write an explanatory note on queueing system.

- (h) Obtain the mean number of units in M/M/1 queueing model with finite system capacity.

4. Answer **any three** of the following questions: $10 \times 3 = 30$

- (a) Prove that

- (i) The p.g.f. $A(s)$ of the marginal distribution of X is given by $A(s) = P(s, 1)$

- (ii) The p.g.f. $B(s)$ of Y is given by $B(s) = P(1, s)$

- (iii) The p.g.f. of $(X+Y)$ is given by $P(s, s)$

$$3+3+4=10$$

- (b) (i) Write a short note on graphical representation of Markov chain.

5



- (ii) Consider two brands of tooth paste which are in competition with each other. Let one brand be represented by 0 and the other be represented by 1. Let 'q' be the probability that an individual using a particular brand in the n th year uses the same brand next year, while 'p' is the probability that he changes the brand, where $p + q = 1$. Write down the transition probability matrix of the Markov chain. Find what will happen in distant future ?

1+4=5

- (c) (i) State and prove the Chapman-Kolmogorov equations.

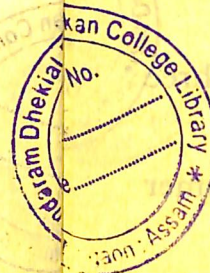
1+5=6

- (ii) Define the following states of Markov chain :
Persistent state, transient state, absorbing state, aperiodic state.

1+1+1+1=4

- (d) (i) Prove that, in an irreducible chain, all the states are of the same type. They are either all transient, all persistent null or all persistent non-null. All the states are aperiodic and in the latter case they all have the same period.

5



- (ii) Consider a Markov chain having state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 1/2 \end{pmatrix}$$

show that all the states of the chain are ergodic.

5

- (e) (i) If $\{N(t)\}$ is a Poisson process and $s < t$, then prove that

$$P_r \{N(s) = k / N(t) = n\} = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

5

- (ii) Prove that the interval between two successive occurrences of Poisson process $\{N(t), t \geq 0\}$ having parameter λ has a negative exponential distribution with mean

$$\frac{1}{\lambda}$$

5

- (f) Under the postulates for Poisson process, prove that $N(t)$ follows Poisson distribution with mean λt i.e. $p_n(t)$ is given by the Poisson law

$$p_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \dots$$

- (g) What do you mean by M/M/1 queueing model with infinite system capacity? Derive the probability distribution of number of customers in this model.

$$3+7=10$$

- (h) The arrivals at a counter in a bank occur in accordance with Poisson process at an average rate of 8 per hour. The duration of service of customer has exponential distribution with a mean of 6 minutes. Find the following :

- (i) the probability that an arriving customer has to wait,
- (ii) the probability that there are three customers in the system,
- (iii) the average number of customer in the queue,

- (iv) the average waiting time in the queue,

- (v) the probability that an arriving customer has to spend less than 15 minutes in the bank.

$$2+2+2+2+2=10$$

